Investing in Uncertainty: A Mathematical Model

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Introduction

- Introduced in 2005, the ETS (Emissions Trading Scheme) is a system in which CO_2 permits are traded with the expectation of producing a cleaner enviorment. The key aspect of this is that higher taxation rates, which are assumed to increase with time, are imposed on companies which are deemed to be heavy emitters of CO_2. To avoid this increased expenditure, capturing and permanently storing the carbon gases is under consideration by CO_2 emitting industries. This method has been coined Carbon Capture and Storage (CCS).

Objective(s):

- **We present an analytically tractable mathematical model which describes the optimal time to invest in CCS technology. We assume that the optimal investment time is primarily driven by the CO_2 permit price which is allowed to vary both deterministically and stochastically.**
- **We assume the stochastic differential equation which governs the carbon permit prices follows a geometric Brownian motion process. With this, we define and solve a set of partial differential equation to obtain to optimal time to invest.**

Optimal Investment

The optimal time to invest in CCS technology is given by a time *T* which maximizes the Net Present Value (NPV) of this investment option [1]:

$$W(C)=\int_0^T P_o(C(t))e^{-rt}\,dt+\int_T^\infty P_n(C(t))e^{-rt}\,dt-I(T)e^{-rT}$$

where P_o and P_n denote the profit functions before and after the retro-fitting of the CCS technology respectively, *r* is the discount rate, *I(T)* is the time-dependent investment cost and *C(t)* is the projected carbon permit price.

Deterministic Carbon Permit Price

Assuming that the time-dependent evolution of *C(t)* is deterministically defined, we can apply standard calculus techniques to determine the maximum of *W(C)* wrt *T.* This occurs when

$$P_n(C(T))-P_o(C(T))=rI(T)-I'(T).$$

To show that *W(C)* is maximized at *T,* we must have

$$\frac{d^2W}{dT}(T)=\Bigg(-r\underbrace{\frac{dW}{dT}(T)}_{(=0)}+\Bigg(\underbrace{\frac{dP_o}{dC}}_{(<0)}-\frac{dP_n}{dC}\Bigg)\underbrace{\frac{dC}{dT}(T)}_{(>0)}-\underbrace{I''(T)+rI'(T)}_{(<0)}\Bigg)e^{-rT}<0$$

where we have assumed that the one-off investment payment is a decreasing convex function. We have also assumed that the profits generated from installing the CCS technology accumulates at a slower rate when compared to profits generated before the CCS installation.

Stochastic Carbon Permit Price

Suppose that *C(t)* is now defined as a stochastic variable which follow a geometric Brownian process. Hence we can write *C(t)* as

$$dC=\mu C dt+\sigma C dz$$

where μ is the drift rate, σ represents the volatility and *dz* is an increment of a Weiner process. As *C(t)* is no longer deterministic, the NPV of the investment option, *W(C),* is given in terms of the conditional expectation of future profits observed at *t*=0

$$W(C)=\mathrm{E}_0\left(\int_0^T P_o(C(t))e^{-rt}\,dt+\int_T^\infty\big(P_n(C(t))+I'(t)-rI(t)\big)e^{-rt}\,dt\right).$$

Applying the Hamilton-Jacobi-Bellman equation yields a set of partial differential equations for the before and after investment regions, which are given by

$$\begin{cases}\text{Before } T & \dfrac{\sigma^2C^2}{2}\dfrac{\partial^2W}{\partial C^2}+\mu C\dfrac{\partial W}{\partial C}-rW_o=-P_o\\[2ex] \text{After } T & \dfrac{\sigma^2C^2}{2}\dfrac{\partial^2W}{\partial C^2}+\mu C\dfrac{\partial W}{\partial C}-rW_n=-P_n+(\xi+r)I(t)\end{cases}$$

As *C(t)* is stochastic, we can no longer explicitly find the optimal time to invest but we can determine a region when the carbon permit price exceeds a certain threshold value, C^*, as seen in Fig.1. This is called a *free-boundary problem*. Assuming a smooth transition from the before into the after investment regions, we find it is optimal to invest when $C(t)>C^*$ and C^* is determined by

$$C^*=\frac{m}{m-1}\left(\frac{\alpha_o-\alpha_n-I'(t)}{r}+I(t)\right)\frac{r-\mu}{q_0-q_n}$$

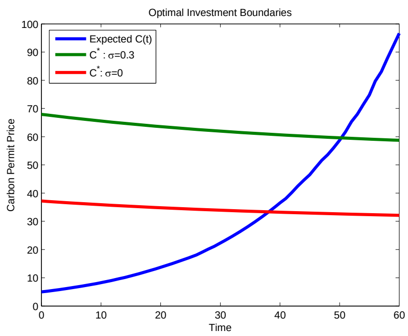

Fig.1: shows the optimal investment boundaries. The blue curve is the expected carbon permit price with μ=0.05. The green curve represents the *free-boundary* with σ=0.3 and the red curve corresponds to the case with σ=0. It is optimal to invest when the expected carbon permit price exceeds the *free boundaries*, i.e with σ=0.3 the optimal time is T≈50 yrs and with σ=0, the optimal time is T≈48 yrs.

Conclusions

- **We have illustrated a means to determine when it is optimal to invest in new CCS technology. If there is no uncertainty in the behaviour of projected carbon permit prices then the optimal investment time can be uniquely determined. The analytic result hold for a broad spectrum of profit and investment-cost functions, provided *I(t)* remains a decreasing convex function. However, if volatility is associated with the permit the uncertainty in the decision-making process delays the optimal time to invest.**

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References

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